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A Sufficient Condition for Topological Chaos with an Application to a Model of Endogenous Growth

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This paper provides an easily verifiable sufficient condition for topological chaos for unimodal maps which can be satisfied when the well-known Li–Yorke condition is not satisfied. It then shows how this result can be applied to a model of endogenous growth with externalities to establish the existence of chaotic equilibrium growth paths in that framework. *Journal of Economic Literature* Classification Numbers: C61, D90, O41. © 2001 Academic Press

Key Words: topological chaos; endogenous growth; externalities; equilibrium dynamics; transversality condition.

1. INTRODUCTION

In this paper I am concerned with two results. First, I provide an easily verifiable sufficient condition for topological chaos which can be satisfied when the well-known Li–Yorke [9] condition is not satisfied. Second, I apply this result to a model of endogenous growth with externalities studied by Boldrin *et al.* [5] (referred to as BNSY [5] below) and show how it can be used to establish the existence of chaotic equilibrium paths in that framework. Let me elaborate somewhat on each result below.

For one-dimensional dynamical systems, the Li–Yorke criterion (equivalently, the condition that there exists a period-three cycle) is a *sufficient* condition for topological chaos (see Section 2 for the relevant definitions). While it is by no means *necessary*, it is widely used to exhibit topological chaos because it remains the easiest criterion to verify. However, for applications in which one knows that the dynamical system *does not* have a period-three cycle, it is not clear what easily verifiable criterion one could check to show the existence of topological chaos.

With this difficulty in applications in mind, we develop below (in Section 2b) a sufficient condition for topological chaos (Proposition 2.3), when the

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law of motion of the dynamical system is a unimodal map (which figures prominently in the mathematical literature on chaos in one dimension). For such maps, it is easy to calculate the (unique interior) fixed point, and it is also fairly straightforward to compute the (first three) iterates of the modal point (the point at which the maximum is attained) of the map. Our criterion is phrased in terms of only these values and should be simple to verify.

The paper [5] develops a model of endogenous growth with externalities, in which the relevant Ramsey–Euler conditions (together with the equilibrium-under-externality condition) yield a first-order nonlinear difference equation in the growth rate of capital. Thus, if there is an equilibrium path, then its growth rate must obey this difference equation. However, a solution path of the difference equation need not yield an equilibrium path, unless a transversality condition can be verified. Thus a central task in their paper is to make sure that the paths generated by the dynamical system satisfy the transversality condition and are therefore actually *equilibrium* paths.

In general, the transversality condition, being an asymptotic condition, cannot be checked *numerically* (using a computer, for example). To elaborate, it can be checked numerically on periodic paths (although for paths with very long periods one can run into memory problems on a computer). It may even be checked on paths which are asymptotically periodic (by using a combination of numerical and obvious analytic methods). However, it cannot be checked numerically on paths which are asymptotically aperiodic, because the “transversality term” will have to be calculated for an infinity of periods. Since asymptotically aperiodic behavior is the essence of chaotic paths, the condition cannot be checked numerically to establish the existence of *chaotic equilibrium* paths.

BNSY [5] report values of the relevant transversality term for long (but finite) time periods. These values *suggest* that the transversality condition ought to be satisfied. I verify analytically that the transversality condition is satisfied, thereby validating their simulation results.

More precisely, I establish conditions on the values of the parameters of this endogenous growth model under which (i) every solution of the (above-mentioned) nonlinear difference equation corresponds to an equilibrium and, simultaneously, (ii) the dynamical system exhibits topological chaos.

Of these, the claim in (ii) above is verified by showing that, under these restrictions on the parameter values, the result of Proposition 2.3 can be directly applied.

Establishing the claim in (i) above is harder. Basically, one has to obtain some “control” over the behavior of the product of the relevant “growth factor” ($\lambda_t^{\eta+\alpha}$) in relation to the discount factor (δ) as the time period (t) becomes large. This, in turn, entails keeping track of “return times,” that is, the time spent in the region where the growth factor exceeds the reciprocal of the discount factor, before it returns to the region where the growth factor is smaller than the reciprocal of the discount factor, and the behavior of the *product* of terms $[\delta\lambda_t^{\eta+\alpha}]$ in each of these regions over these time intervals.

2. TOPOLOGICAL CHAOS

2a. *Definitions*

Let X be a compact interval in $\mathbb{R}$, the set of reals. Let $f\colon X \to X$ be a continuous map of the interval X into itself. The pair (X, f) is called a *dynamical system*; X is called the *state space* and f the *law of motion* of the dynamical system.

We write $f^0(x) = x$ and, for any integer $n \geqslant 1$, $f^n(x) = f[f^{n-1}(x)]$. If $x \in X$, the sequence $\tau(x) = \{f^n(x)\}_0^\infty$ is called the *trajectory* from (the initial condition) x. The *orbit* from x is the set $\gamma(x) = \{y\colon y = f^n(x)$ for some $n \geqslant 0\}$.

A point $x \in X$ is a *fixed point* of f if $f(x) = x$. A point $x \in X$ is called a *periodic point* of f if there is $k \geqslant 1$ such that $f^k(x) = x$. The smallest such k is called the *period* of x. (In particular, if $x \in X$ is a fixed point of f it is periodic with period 1). If $x \in X$ is a periodic point with period k, we also say that the orbit of x (or trajectory from x) is periodic with period k.

The dynamical system (X, f) is called *turbulent*[2] (see Fig. 1) if there exist points a, b, c in X such that

$$f(b) = f(a) = a, \quad f(c) = b, \qquad \text{and either} \quad a < c < b \quad \text{or} \quad b < c < a. \tag{2.1}$$

To study the nature of trajectories which are not periodic, we define a "scrambled" set. A set $S \subset X$ is called a *scrambled set* if it possesses the following two properties:

(i) If $x, y \in S$ with $x \neq y$, then

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0 \qquad \text{and} \qquad \liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0.$$

[2] The concept of "turbulence" has developed from the discussions of Ruelle and Takens [12] and Lasota and Yorke [8]. The concept is also referred to in the literature as a "2-horseshoe" (see Alseda *et al.* [2]). Block and Coppel [4, p. 25] call (X, f) turbulent if there exist compact subintervals J, K with at most one common point such that $J \cup K \subset f(J) \cap f(K)$. They show that this definition is equivalent to the more "operational" definition given in (2.1).

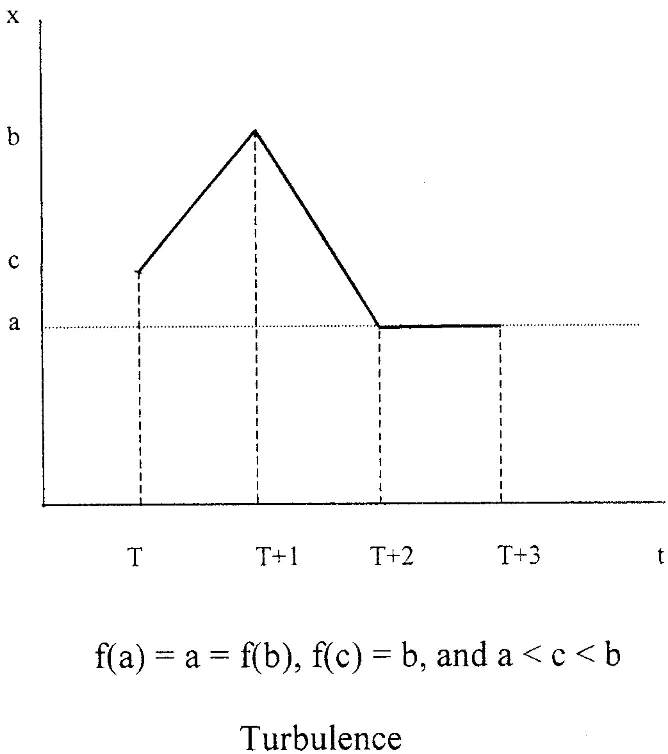


FIGURE 1

(ii) If $x \in S$ and y is any periodic point of f,

$$\limsup_{n \to \infty} |f^n(x) - f^n(y)| > 0.$$

Thus trajectories starting from points in a scrambled set are not even "asymptotically periodic." Furthermore, for any pair of initial states in the scrambled set, the trajectories move apart and return close to each other infinitely often.

A finite set $E \subset X$ is called (n, ε)-*separated* ($n = 1, 2, ...$ and $\varepsilon > 0$) if, for every $x, y \in E$, $x \neq y$, there is $0 \leqslant k < n$ such that $|f^k(x) - f^k(y)| \geqslant \varepsilon$. Let $s(n, \varepsilon)$ denote the maximal cardinality of an (n, ε)-separated set. We define

$$\psi_\varepsilon(f, X) = \limsup_{n \to \infty} (1/n) \log s(n, \varepsilon)$$

and the topological entropy[3] of f as

$$\psi(f, X) = \lim_{\varepsilon \to 0} \psi_\varepsilon(f, X).$$

We will say that the dynamical system (X, f) exhibits *topological chaos* if f has positive topological entropy.

2b. *A Sufficient Condition for Topological Chaos*

Given our definition of topological chaos,[4] it would be useful to know how it can be verified, given a dynamical system (X, f). The following characterization result, due to Misiurewicz [11], is extremely useful in this respect, as it relates positive topological entropy of f to the existence of certain periodic points of (X, f).

THEOREM 2.1. *The dynamical system (X, f) has positive topological entropy if and only if it has a periodic point of period that is not a power of* 2.

In view of the above result, the well-known Li–Yorke theorem (stated below) can be regarded as providing a sufficient condition for the existence of topological chaos.

PROPOSITION 2.1. *Assume that there is some point x^* in X such that*

$$f^3(x^*) \leqslant x^* < f(x^*) < f^2(x^*) \qquad (\text{or } f^3(x^*) \geqslant x^* > f(x^*) > f^2(x^*)). \quad (2.2)$$

Then

(i) *for every positive integer $k = 1, 2, \ldots$, there is a periodic point of period k.*

(ii) *there is an uncountable scrambled set $S \subset X$.*

The Li–Yorke criterion (2.2) is particularly appealing, because it is an easy condition to verify. (See Fig. 2.) It can be checked that (X, f) satisfies (2.2) if

[3] The formal definition of topological entropy was given by Adler *et al.* [1]. Bowen [6] provided the more "operational" definition which we use here. In our context, the two definitions are equivalent; see Bowen [7] for a proof.

[4] It is known (see, for example, [4, Proposition 27, p. 143]) that if (X, f) has a cycle of period that is not a power of 2, then it has an uncountable scrambled set. Thus, in view of Theorem 2.1, if (X, f) exhibits topological chaos (that is, f has positive topological entropy), then it has an uncountable scrambled set. However, it is also known (see [4, Example 29, p. 146 and Corollary 26, p. 142]) that a dynamical system (X, f) can have an uncountable scrambled set, even though the topological entropy of f is zero. This is the reason why one should not define topological chaos in terms of an uncountable scrambled set.

and only if (X, f) has a periodic point of period three. But, if we know that a dynamical system *does not* satisfy (2.2) (equivalently, does not have a period-three cycle), we do not know what simple criterion to check to show topological chaos.[5]

We develop in this section a sufficient condition for topological chaos when the law of motion of the dynamical system is a unimodal map. For such maps, it is typically easy to calculate the (unique interior) fixed point and the (first three) iterates of the modal point. Our condition is in terms of only these values and should be easily verifiable.

Our result is based on the theory of turbulent dynamical systems.[6] Suppose (X, f) is turbulent in the sense of (2.1). Consider the case where $a<c<b$. Since $f(a)=a<c<b=f(c)$, there exists a point $q \in (a, c)$ such that $f(q)=c$. Then $b=f^2(q)$ and $a=f^3(q)$, so that $f^3(q)<q<f(q)<f^2(q)$. Thus, (X, f) has a period-three cycle by Proposition 2.1. We can summarize this as

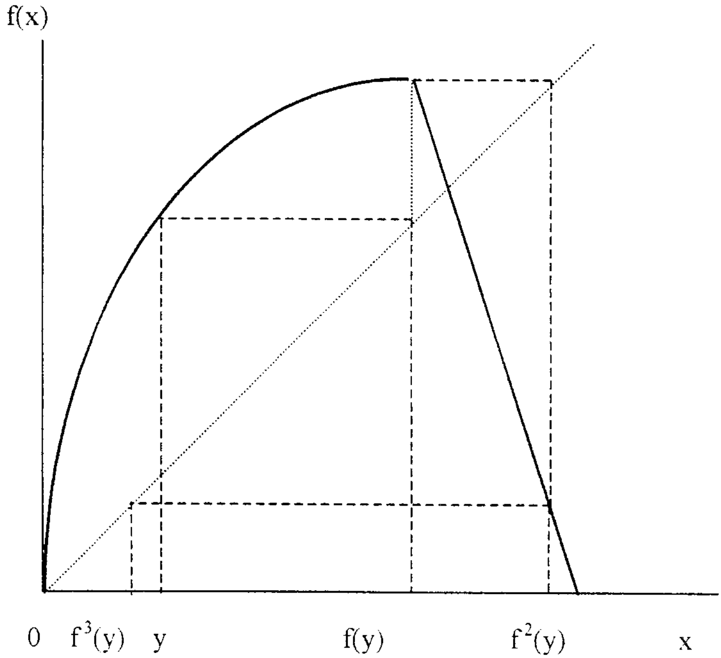
PROPOSITION 2.2. *Let (X, f) be a dynamical system, which is turbulent in the sense of* (2.1). *Then (X, f) has a period-three cycle.*

Given Proposition 2.2, our strategy is to look for sufficient conditions on (X, f) under which the dynamical system (X, f^2) is turbulent. For if (X, f^2) is turbulent, then (X, f) has a period-three cycle or a period-six cycle; and then, by Theorem 2.1, (X, f) has positive topological entropy and therefore exhibits topological chaos.

We will be concerned with a dynamical system (X, f), where the state space, X, will be an interval $[a, b]$ of the real line, with $0 \leqslant a < b < \infty$. The law of motion, f, will be a continuous function from X to X, with the following properties:

[5] In seeking weaker sufficient conditions for topological chaos of (X, f) than the Li–Yorke condition, one might proceed as follows. Given the Sarkovskii theorem (see [13, 3, and 4]), one might try to obtain sufficient conditions to ensure that cycles of periods 5, 7, 9, 11, ... (odd periods greater than three) occurred or, perhaps, to ensure that cycles of periods 6, 10, 14, ... (periods of the form $2q$ where q is odd and greater than 1) occurred, and so on. Given that we are seeking sufficient conditions which are *easy to check*, the most promising are cycles of period 5 or 6. However, unlike the Li–Yorke condition which involves only three iterates of f, these involve five or six iterates of f.

[6] The reason for focusing on the concept of turbulence in seeking weaker sufficient conditions for topological chaos of (X, f) than the Li–Yorke condition is that if (X, f) has a cycle of *any* odd period greater than 1, then (X, f^2) is turbulent; while, if (X, f^2) is turbulent, then (X, f) has a period-6 cycle. That is, in terms of the Sarkovskii order, the turbulence of (X, f^2) sits between the cycles of (X, f) of odd period greater than 1 and the cycle of (X, f) of period 6. Thus, obtaining a sufficient condition for the turbulence of (X, f^2) is more appealing than that for a period-five cycle, and "almost as appealing" as that for a period-six cycle, provided the condition is easy to check (relative to a period five or six cycle). It turns out that we can check the turbulence of (X, f^2) by computing the interior fixed point of f and just three iterates of the map f (*not* f^2) of the modal point.



$f^3(y) < y < f(y) < f^2(y)$

Li-Yorke Condition

FIGURE 2

(i) There is m in (a, b), with f strict increasing on $[a, m]$ and strictly decreasing on $[m, b]$.

(ii) $f(a) \geqslant a$, $f(b) < b$, and $f(x) > x$ for all x in $(a, m]$.

Define $\mathbf{F} = \{f\colon\ f$ is a continuous map from X to X, satisfying (i) and (ii) above$\}$.

Note that $g(x) = [f(x) - x]$ is continuous and strictly decreasing on $[m, b]$, with $g(m) > 0$ and $g(b) < 0$. Thus, there is a unique value of x in (m, b), call it z, such that $g(z) = 0$; that is, z is an interior fixed point of f. Note that, by (ii) above, $f(x) > x$ for all x in $(a, m]$, so there is no other interior fixed point of f. (If $f(a) = a$, then a is also a fixed point of f, but it is at the boundary. If $f(a) > a$, then z is the only fixed point of f.)

We now look at the iterates of the modal value, m, under iterates of f. The first iterate is easy to analyze. Clearly, by (ii) above $f(m) > m$, and since f maps from X to X, $f(m) \leqslant b$. Since f is decreasing on $[m, b]$ and z is in (m, b), we have $f(m) > f(z)$. To summarize, we have

$$m < z = f(z) < f(m) \leqslant b. \tag{2.3}$$

Consider now the second iterate of m, under f; that is, $f^2(m)$. Since f is strictly decreasing on $[m, b]$, and $f(z)$ and $f(m)$ are both in $[m, b]$ with $f(z) < f(m)$, we must have $f^2(z) > f^2(m)$. That is, we have $f^2(m) < z$, which can be combined with (2.3) to give us

$$f^2(m) < z = f(z) < f(m) \leqslant b. \tag{2.4}$$

Note that, in general, we cannot order the points m and $f^2(m)$. Therefore, we analyze two cases (depending on whether $f^2(m) \geqslant m$ or $f^2(m) < m$) in turn.

In the case where $f^2(m) \geqslant m$, it is easy to see that the long-run dynamics starting from any x in (a, b) is confined to the interval $[m, f(m)]$. And, in this interval, we have f strictly decreasing. Thus, all trajectories must converge to either the fixed point, z, or to a two-period cycle.

Consequently, the dynamics in this case cannot be chaotic.

Thus, a *necessary* condition for chaotic dynamics is

$$f^2(m) < m. \tag{2.5}$$

Consequently, we maintain (2.5) in what follows. Combining (2.3), (2.4), and (2.5), we have the following ordering of the points considered so far:

$$f^2(m) < m < z = f(z) < f(m) \leqslant b. \tag{2.6}$$

We are now in a position to look at the third iterate of m under the map f, that is, $f^3(m)$. There are two pieces of information we can deduce about this value. First, since $f^2(m)$ is in the interval $[a, m)$ and f is increasing on $[a, m]$, we have $f^3(m) < f(m)$. Second, since $f(x) \geqslant x$ on $[a, m]$, we also have $f^3(m) \geqslant f^2(m)$. To summarize, we have

$$f^2(m) \leqslant f^3(m) < f(m). \tag{2.7}$$

In general, we cannot order the points $f^3(m)$ and m. If it turned out that $f^3(m) \leqslant m$, then we must have a period-three cycle. To see this, note first that if $f^3(m) = m$, then $(m, f(m), f^2(m))$ is a period-three cycle [using (2.5)]. Second, if $f^3(m) < m$, then we have $f(f^2(m)) < m$, while $f(m) > m$, so there is some value, q, in $(f^2(m), m)$ such that $f(q) = m$. Then we have

$$f^3(q) = f^2(m) < q < m = f(q) < f(m) = f^2(q), \tag{2.8}$$

so that the Li–Yorke condition (2.2) is satisfied and we have a period-three cycle.

We now show that a satisfactory theory can be developed under the weaker condition

$$f^3(m) < z. \tag{2.9}$$

In words, the condition is that the third iterate of the modal point (m) under the map f is less than the interior fixed point (z). This is weaker than assuming $f^3(m) \leqslant m$, since $m < z$, by (2.3).

Let us state clearly what we will show below. Under (i), (ii), the *necessary* condition of chaos (that is, (2.5)), and Condition (2.9), we will demonstrate that (X, f^2) is turbulent.

We define, sequentially, four values, P, Q, R and S in $[a, b]$, and using those we verify that the dynamical system (X, f^2) is turbulent. Denote the restriction of f to $[a, m]$ by U; then U is an increasing function. Denote the restriction of f to $[m, b]$ by V; then, V is a decreasing function.

By (2.5), we have $f^2(m) < m$, so that $f^2(m)$ belongs to $[a, m)$. By (2.9), we have $f(f^2(m)) < z$, so we have $U(f^2(m)) < z$. Also, $f(m) > z$ by (2.3), so that $U(m) > z$. Thus, there is some P in $(f^2(m), m)$ such that $f(P) = U(P) = z$.

By (2.3), we have $m < z < f(m) \leqslant b$, so that z and $f(m)$ both belong to $[m, b]$. Thus, $V(z) = f(z) = z > m > P$, and $V(f(m)) = f^2(m) < P$. So, there is some value Q in $(z, f(m))$ such that $f(Q) = V(Q) = P$.

As noted above, z and $f(m)$ both belong to $[m, b]$, with $z < f(m)$. So, by (2.3), we have $V(z) = f(z) = z > m$. Also, by (2.5), $V(f(m)) = f^2(m) < m$. Thus, there is some R in $(z, f(m))$ such that $f(R) = V(R) = m$. Since $V(Q) = P < m = V(R)$ and V is decreasing, we must have $Q > R$.

Note, finally, that $f^2(R) = f(m) > Q$ and $f^2(z) = z < Q$. Thus, there is some S in (z, R) such that $f^2(S) = Q$. We can summarize the ordering of all the relevant values as follows:

$$a \leqslant f^2(m) < P < m < z < S < R < Q < f(m) \leqslant b. \tag{2.10}$$

Furthermore, the values are related by the equations

$$f(Q) = P, \quad f(P) = z, \quad f(R) = m, \qquad \text{and} \quad f^2(S) = Q. \tag{2.11}$$

Define $h = f^2$; then h is a continuous map from X to X. Also, we have the triple of values, (z, S, Q), satisfying (i) $z < S < Q$ (using (2.10)) and (ii) $h(Q) = z = h(z)$ and $h(S) = Q$ (using (2.11)). Thus, the dynamical system (X, h) is turbulent.

We summarize our result, based on the above analysis, as follows.

PROPOSITION 2.3. *Let* (X, f) *be a dynamical system, with* $X = [a, b] \subset \mathbb{R}$, *where* $0 \leqslant a < b < \infty$ *and* $f \in \mathbf{F}$. *If* f *satisfies* $f^2(m) < m$ *and* $f^3(m) < z$ *then* (X, f^2) *is turbulent and* (X, f) *exhibits topological chaos.*

2c. *An Example*

This example is based on a model of growth through cycles due to Matsuyama [10]. The state space, Y, is the set of nonnegative reals, and the law of motion, ϕ, is described by two parameters, G and σ, both of which belong to $(1, \infty)$. Following Matsuyama, it is convenient to define the parameter θ as follows: $\theta = (1 - (1/\sigma))^{1-\sigma}$. Thus, as σ varies from 1 to ∞, θ varies from 1 to e. The law of motion, ϕ, can be defined in terms of G, σ, and θ as

$$\phi(k) = Gk^{1-(1/\sigma)} \qquad \text{for } k \text{ in } [0, 1]$$

$$\phi(k) = Gk/[1 + \theta(k-1)] \qquad \text{for } k \text{ in } (1, \infty).$$

Thus, the map, ϕ, is "tent-like," with a maximum and a kink at $k = 1$, increasing on $[0, 1]$, and decreasing on $(1, \infty)$. (See Fig. 3.) The interesting phenomenon of growth with fluctuations arises when the parameters satisfy the restriction $1 < G < \theta - 1$, so we maintain this restriction in what follows.

We note that $\phi(1) = G$ and define $H = \phi(G) = G^2/[1 + \theta(G-1)]$. Then, given the restriction $1 < G < \theta - 1$, we have $H < 1$. The interval $X = [0, G]$ is an invariant set in the sense that $\phi(X) \subset X$. Thus, if k is once in X, then all iterates of k (under the map ϕ) are in X. And, if k is in (G, ∞), then in the next time period it is in X. Thus, all the long-run dynamics are confined to the interval X, the other states in Y being transient. Define f to be the restriction of ϕ to X; then (X, f) is a dynamical system and we can study whether this generates topological chaos.

The dynamical system (X, f) satisfies conditions (i) and (ii) of Section 2b, with the modal point $m = 1$ and $G > 1$. Further, as verified above when $G < \theta - 1$, we have $\phi(G) < 1$, so that $\phi^2(1) < 1 = m$, and the necessary condition of chaos (given by (2.5) in Section 2b) is also satisfied. Matsuyama [10] notes that (X, f) *cannot* have a period-three cycle. We show, by verifying condition (2.9) of Section 2b that, nevertheless, (X, f) can exhibit topological chaos.

The unique interior fixed point of ϕ, call it z, is easy to calculate. It satisfies the equation

$$Gz/[1 + \theta(z-1)] = z, \tag{2.12}$$

so that $z = 1 + [(G-1)/\theta]$. Also, we have $\phi(1) = G$ and $\phi^2(1) = \phi(G) = G^2/[1+\theta(G-1)]$. So, we get $\phi^3(1) = G(G^2/[1+\theta(G-1)])^{1-(1/\sigma)}$. Thus, condition (2.9) reduces to the inequality

$$G(G^2/[1+\theta(G-1)])^{1-(1/\sigma)} < 1 + [(G-1)/\theta]. \tag{2.13}$$

It can be verified that (2.13) holds if σ is sufficiently large (θ is sufficiently close to e), and simultaneously G is sufficiently close to 1. We provide here a numerical example. Choose $\sigma = 50$ and $G = 1.01$. Then $(1-(1/\sigma)) = 0.98$ and $\theta = 1/(0.98)^{49} = 2.6910532$. With these values, the left-hand side expression in (2.13) is 1.0034352, while the right-hand side expression in (2.13) is 1.003716. Thus the inequality in (2.13) does hold for these values of the parameters (see Fig. 3), and (X, f) exhibits topological chaos by Proposition 2.3.

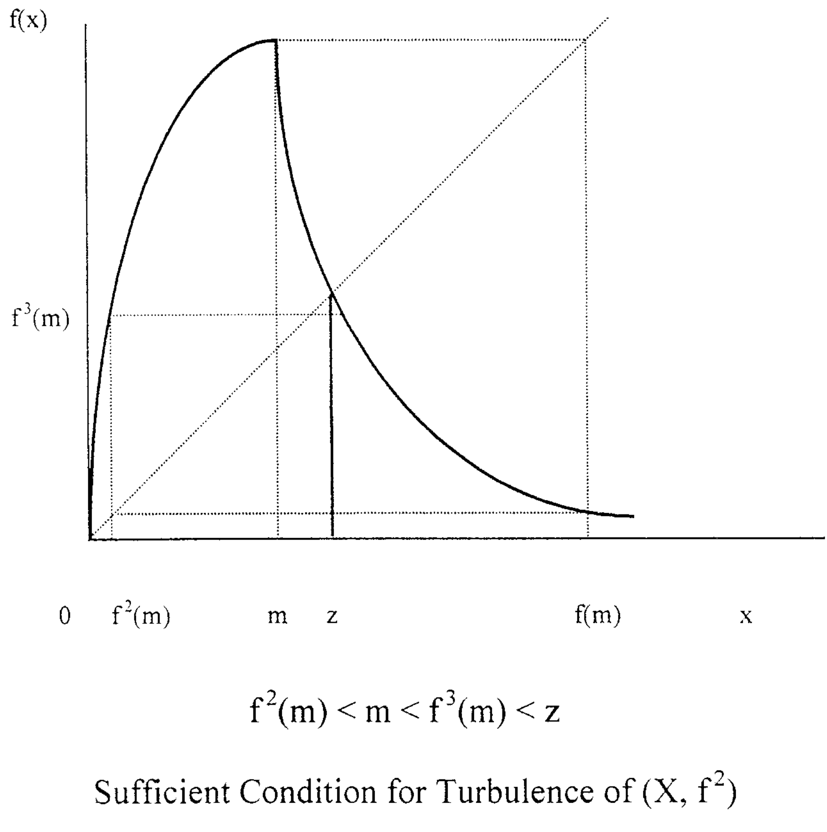


FIGURE 3

3. BASIC PROPERTIES OF AN ENDOGENOUS GROWTH MODEL

Our model is the two-sector model with externality that is used in [5]. There are two goods produced in two different sectors, the consumption good and the investment good. Let c_t and I_t be the amount of the consumption good and that of the investment good that are produced in period t. The consumption good is produced from both capital K_{1t} and labor L. The investment good is produced from capital K_{2t} alone. Let k_t be the amount of the capital good that is available at time t. Then

$$K_{1t} + K_{2t} = k_t. \tag{3.1}$$

The initial amount of capital input is given by $k_0 = \bar{k} > 0$. The economy is also endowed with a fixed amount of labor $L = 1$.

Externalities affect the production of the consumption good. Let e_t denote the magnitude of this externality. The production function of the consumption good sector (Sector 1) is

$$c_t = e_t^{\bar{\eta}} K_{1t}^{\bar{\alpha}} L^{1-\bar{\alpha}}, \tag{3.2}$$

where $0 < \bar{\alpha} < 1$ and $\bar{\eta} > 0$. The production function of the investment good sector (Sector 2) is

$$I_t = \theta K_{2t}, \tag{3.3}$$

where $\theta > 1$. We assume full depreciation of the capital good, so that

$$k_{t+1} = I_t. \tag{3.4}$$

Let u denote the utility function of the representative consumer, and assume that

$$u(c) = c^{1-\sigma} \qquad \text{where} \quad 0 < \sigma < 1. \tag{3.5}$$

Denote by δ, $0 < \delta < 1$, the discount factor of future utilities. Then, the representative agent solves the following optimization problem. Given a sequence of externalities $\{e_t\}_{t=0}^{\infty}$,

$$\max_{\{c_1, k_t, K_{1t}, K_{2t}, I_t\}_{t \geq 0}} \sum_{t=0}^{\infty} \delta^t u(c_t), \tag{3.6}$$

such that

$$c_t = e_t^{\bar{\eta}} K_{1t}^{\bar{\alpha}}, \qquad I_t = bK_{2t}, \qquad K_{1t} + K_{2t} = k_t,$$

$$k_{t+1} = I_t \qquad \text{for} \quad t \geq 0, \quad \text{and} \quad k_0 = \bar{k}.$$

The level of the externality generated in period t is equal to the amount of the capital good employed in that period by the two sectors; that is,

$$e_t = k_t. \tag{3.7}$$

Since $c_t = e_t^{\bar{\eta}}[(\theta k_t - k_{t+1})/b]^{\bar{\alpha}}$, the above optimization problem (3.6) gives rise to

$$\max_{\{k_t\}_{t\geqslant 0}} \sum_{t=0}^{\infty} \delta^t e_t^{\eta}(\theta k_t - k_{t+1})^{\alpha}$$
$$\text{s.t.} \quad k_0 = \bar{k} \quad \text{and} \quad 0 \leqslant k_{t+1} \leqslant \theta k_t, \quad \text{for} \quad t \geqslant 0, \tag{3.8}$$

where $\eta = (1-\sigma)\,\bar{\eta}$ and $\alpha = (1-\sigma)\,\bar{\alpha}$. We refer to a path of accumulation $\{k_t\}_0^{\infty}$ as an *equilibrium path* if

(i) it solves the optimization problem (3.8) and
(ii) $e_t = k_t$ for $t = 0, 1, 2, \ldots$.

We assume in what follows that $0 < \alpha < 1$ and $\eta > (1-\alpha)$.

We call a path $\{k_t\}_0^{\infty}$ satisfying $k_0 = \bar{k}$ and $0 < k_{t+1} < \theta k_t$ an *interior path* (from $k_0 = \bar{k}$). We call an *interior path* $\{k_t\}_0^{\infty}$ a *balanced growth path* if it is in equilibrium and $k_{t+1}/k_t = \lambda$ for $t = 0, 1, \ldots$. For an interior path $\{k_t\}_0^{\infty}$, define $\lambda_t = (k_{t+1}/k_t)$ for $t \geqslant 0$; then, $0 < \lambda_t < \theta$ for $t \geqslant 0$.

An interior path $\{k_t\}_0^{\infty}$ satisfies the Ramsey–Euler equations if

$$e_t^{\eta}(\theta k_t - k_{t+1})^{\alpha-1} = \delta\theta e_{t+1}^{\eta}(\theta k_{t+1} - k_{t+2})^{\alpha-1} \qquad \text{for} \quad t \geqslant 0. \tag{3.9}$$

Thus, an interior path satisfying the Ramsey–Euler equation, and the condition $e_t = k_t$, for $t \geqslant 0$, must satisfy a difference equation in the single variable, λ_t, as follows:

$$(\theta - \lambda_{t+1}) = (\delta\theta)^{[1/(1-\alpha)]}(\theta - \lambda_t)(\lambda_t)^{\{[\eta/(1-\alpha)]-1\}} \qquad \text{for} \quad t \geqslant 0. \tag{3.10}$$

Denoting $(\theta - \lambda_t)$ by z_t for $t \geqslant 0$, (3.10) can be transformed into the recursive system, $z_{t+1} = f(z_t)$, where f is a map from $X = [0, \theta]$ to $\mathbb{R}$, given by

$$f(z) = Az(\theta - z)^{\beta}, \tag{3.11}$$

with $\beta = [\eta/(1-\alpha)] - 1$, $A = (\delta\theta)^{[1/(1-\alpha)]}$. Since $\eta > (1-\alpha)$, we have $\beta > 0$.

It is easy to check that the map, f, is unimodal on X, with $f(0) = f(\theta) = 0$. Specifically, the modal point (the point at which f attains a maximum on X) is given by

$$m = \theta/(1+\beta). \tag{3.12}$$

Furthermore, f is strictly increasing on $[0, m)$ and strictly decreasing on $[m, \theta]$.

The maximum value of the function, $f(m)$, will be strictly less than θ if

$$\delta^{1/(1-\alpha)}\theta^{\beta+[1/(1-\alpha)]} < (\beta+1)^{(\beta+1)}/\beta^{\beta}. \tag{3.13}$$

Under the restriction (3.13), we have $f(z)$ in $(0, \theta)$ whenever z is in $(0, \theta)$. Thus, (X, f) is a dynamical system, with f a map from X to X; further, if z_0 is in $(0, \theta)$, then z_t is in $(0, \theta)$ for all $t \geqslant 0$.

We can ensure that $f(z) > z$ for all z in $(0, m]$ if

$$\delta^{1/(1-\alpha)}\theta^{\beta+[1/(1-\alpha)]} > (\beta+1)^{\beta}/\beta^{\beta}. \tag{3.14}$$

We maintain the restrictions on the parameters expressed in (3.13) and (3.14) in what follows, and we call it Condition 1. (This is a stronger condition than Condition 1 in [5].)

CONDITION 1.

$$(\beta+1)^{\beta}/\beta^{\beta} < \delta^{1/(1-\alpha)}\theta^{\beta+[1/(1-\alpha)]} < (\beta+1)^{(\beta+1)}\ \beta^{\beta}.$$

Since $f(m) > m$, while $f(\theta) = 0 < \theta$, there is some $\bar{z}$ in (m, θ), such that $f(\bar{z}) = \bar{z}$. This $\bar{z}$ is uniquely given by

$$\bar{z} = \theta - (1 - A)^{1/\beta}. \tag{3.15}$$

Then $\bar{z}$ corresponds to a steady-state equilibrium with growth factor $\lambda = \theta - \bar{z}$. One can ensure a growth factor exceeding 1 in such an equilibrium by assuming that $A < 1$. This restriction on the parameters is also maintained in what follows, and we write it as Condition 2. (It corresponds to Condition 2 in [5].)

CONDITION 2.

$$\theta < (1/\delta).$$

4. CHAOTIC EQUILIBRIUM DYNAMICS

In this section, we wish to identify restrictions on parameter values (of the endogenous growth model of Section 3) under which (i) the dynamical system (X, f) exhibits topological chaos and (ii) any sequence $\{z_t\}$ generated by the dynamical system (X, f), starting with z_0 in $(0, \theta)$, corresponds to an equilibrium path.

Given Proposition 2.3 and the properties of (X, f) already verified in Section 3, (X, f) will exhibit topological chaos (verifying (i) above) if the parameter restrictions ensure that

$$f^2(m) < m \qquad \text{and} \quad f^3(m) < \bar{z}. \tag{4.1}$$

To verify (ii) above, we note that the basic relation between sequences generated by the dynamical system (X, f) and equilibrium paths is given by the following result, which is just a restatement of Lemma 2 in [5].

PROPOSITION 4.1. *Let z_0 belong to $(0, \theta)$ and let $\{z_t\}$ be the sequence generated by iterates of f, starting with z_0. Define $k_0 = k$ and $k_{t+1} = (\theta - z_t)\, k_t$ for $t \geqslant 0$. Then $\{k_t\}$ is an equilibrium path from k if*

$$\sum_{t=0}^{\infty} \delta^t k_t^{\eta - \alpha} < \infty. \tag{4.2}$$

The condition (4.2) ensures that both the summability and the transversality conditions are satisfied by $\{k_t\}$, so that it is an equilibrium path from k (by Lemma 1 of [5]).

In order then to show that any sequence $\{z_t\}$ generated by the dynamical system (X, f), starting with z_0 in $(0, \theta)$, corresponds to an equilibrium path, one has to show that the sequence $\{k_t\}$, associated with $\{z_t\}$ as in Proposition 4.1 above, automatically satisfies (4.2), under some suitable parametric restrictions. While such parametric restrictions can be found, the problem in demonstrating *chaotic equilibrium paths* is to find restrictions which will ensure that (4.1) and (4.2) are *simultaneously* satisfied.

We proceed as follows. In order to demonstrate clearly our approach to the problem, we refrain from specifying (ranges of) numerical values to the parameters at present and instead focus on *relationships between the parameters* that make the method work.

To begin, let us *assume* that parameter restrictions have been found such that the following condition holds:

$$f^2(m) < m < f^3(m) < z \tag{E.0}$$

Note that (E.0) implies that (4.1) is satisfied. (We will provide a robust numerical example in Section 5, verifying that this assumption is satisfied.) Now, we try to find (additional) restrictions on the parameters such that (4.2) will also be satisfied.

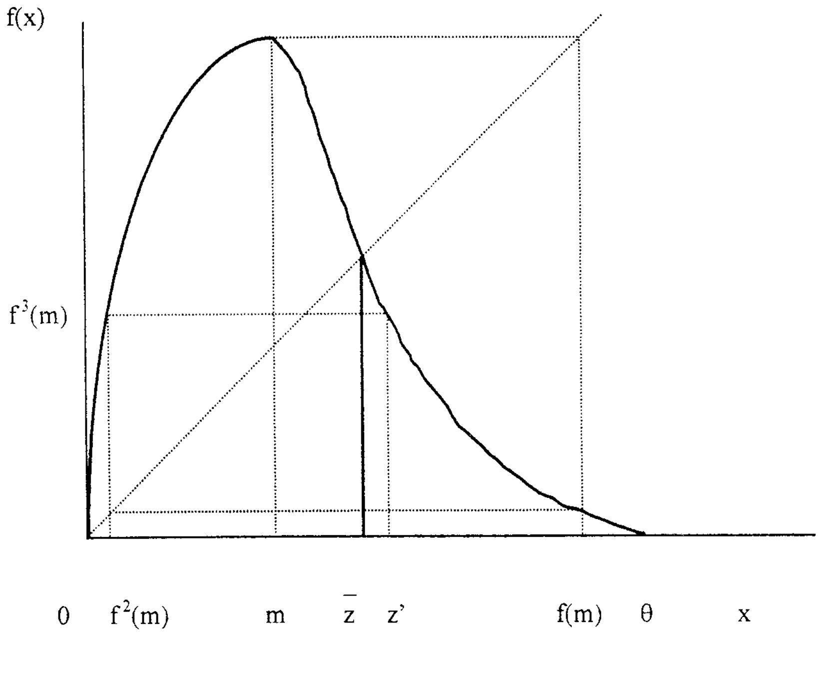
To this end, it is convenient to rewrite (4.2) in terms of the growth factors of the capital stock sequence. For the $\{k_t\}$ sequence (generated by the $\{z_t\}$ sequence, as in Proposition 4.1 above), we have $\lambda_t = (k_{t+1}/k_t)$ and $\lambda_t = \theta - z_t$ for $t \geqslant 0$. Next, let us define, for $t \geqslant 0$, $v_t = \delta^{1/(1-\alpha)} \lambda_t^{\beta + [1/(1-\alpha)]} = (\delta \lambda_t^{\eta+\alpha})^{1/(1-\alpha)}$. Then, for $t \geqslant 1$, we have

$$\delta^t k_t^{\eta+\alpha} = k^{\eta+\alpha} \prod_{s=0}^{t-1} v_s^{(1-\alpha)}. \tag{4.3}$$

This means that in order to verify (4.2), we will need to study the dynamic behavior of the sequence $\{v_t\}$ carefully.

Denote the interval $[f^2(m), f(m)]$ by I. Note that for y in $[f^2(m), m]$ we have (by the inequality (3.14) of Condition 1) $f(y) > y \geqslant f^2(m)$, and for y in $(m, f(m)]$ we have f decreasing in this region, so that $f(y) \geqslant f^2(m)$. Thus, for all y in I, $f(y) \geqslant f^2(m)$. Also, for all y in X, $f(y) \leqslant f(m)$, by definition of m. Thus, for all y in I, we have $f(y)$ in I, and I is an invariant set.

If z_0 is in $(0, f^2(m))$, then z_t will monotonically increase till it enters I, and thereafter it will stay in I. If z_0 is in $(f(m), \theta)$, then $f(z_0)$ is in $(0, f^2(m))$, so that for $t \geqslant 1$, z_t will monotonically increase till it enters I,



Chaotic Equilibrium Dynamics

FIGURE 4

and thereafter it will stay in I. Thus, the long-run dynamics of z_t is confined to the invariant set, I.

We can split up the interval into three subintervals as follows. We have $f(\bar{z}) = \bar{z} > f^3(m)$ and $f(f(m)) < f^3(m)$. So there is z' in $(\bar{z}, f(m))$ such that $f(z') = f^3(m)$. (See Fig. 4). This z' is uniquely defined, since f is strictly decreasing in the region $(\bar{z}, f(m))$. We now define $I_1 = [f^2(m), f^3(m))$, $I_2 = [f^3(m), z')$, and $I_3 = [z', f(m)]$. We also define, for future reference, $J_2 = [f^3(m), z']$ and $J_3 = (z', f(m)]$. Note that I_1, I_2, and I_3 are disjoint intervals, with union I; similarly, I_1, J_2, and J_3 are disjoint intervals with union I.

Now, note that if y is in I_1 or I_2, then $f(y) \geqslant f^3(m)$, so $f(y)$ cannot belong to I_1. Consequently, if for some y' in I, $f(y')$ does belong to I_1, then y' must belong to I_3.

It also follows from this observation that if $\{z_t\}$ is generated by iterates of f, starting with any z_0 in $(0, \theta)$, then there is some time period at which z_t is in $[f^3(m), f(m)]$.

This suggests the following line of analysis. Suppose that we can ensure that the following two conditions are satisfied:

$$\delta^{1/(1-\alpha)}(\theta - f^3(m))^{[1/(1-\alpha)]+\beta} < 1, \tag{E.1}$$

$$\{\delta^{1/(1-\alpha)}(\theta - y)^{[1/(1-\alpha)]+\beta}\}\{\delta^{1/(1-\alpha)}(\theta - f(y))^{[1/(1-\alpha)]+\beta}\} < 1 \quad \text{for all} \quad y \text{ in } I_3. \tag{E.2}$$

Then the condition (4.2) will be satisfied.

Let us explain why this is so. Given (E.1), let us denote $\delta^{1/(1-\alpha)}(\theta - f^3(m))^{[/(1-\alpha)]+\beta}$ by D_1. Given (E.2), the *maximum* of the expression $\{\delta^{1/(1-\alpha)}(\theta - y)^{[1/(1-\alpha)]+\beta}\}\{\delta^{1/(1-\alpha)}(\theta - f(y))^{[1/(1-\alpha)]+\beta}\}$ over the compact interval I_3 is less than 1; denote this value by D_2. Then $D \equiv \max(D_1, D_2) < 1$ and $d = D^{(1-\alpha)} < 1$ also.

We know that the sequence $\{z_t\}$, generated by iterates of f and starting from any z_0 in $(0, \theta)$, will enter $[f^3(m), f(m)]$ for some t. Let T be the first time period when this happens (T can be zero). We now examine the sequence $\{v_t\}$ for $t \geqslant T$, remembering that, since z_T is in I, we have z_t in I for all $t \geqslant T$. Given (E.1), if z_t is in $[f^2(m), f^3(m)]$, then $v_t \leqslant D_1 \leqslant D$, so $v_t^{(1-\alpha)} \leqslant d < 1$. And, if z_t is not in $[f^3(m), f(m)]$, that is, z_t is in $[f^2(m), f^3(m))$, then z_{t-1} is in $(z', f(m)]$, and so by using (E.2) we have $v_t v_{t-1} \leqslant D_2 \leqslant D$ and $(v_t^{(1-\alpha)})(v_{t-1}^{(1-\alpha)}) \leqslant d < 1$. This means that the term in (4.3) is dominated by a sequence which decreases geometrically at the factor $d < 1$. This last statement requires elaboration, which we now provide.

Our objective is to evaluate products of v_t of the form $\prod_{t=T}^{T+N} v_t$, where $N \geqslant 1$. In the set of time periods $\{T, T+1, ..., T+N\}$, let n be the number of periods in which z_t is in I_1 (n can be zero). It follows that for each of these t we have the preimage of z_t (that is, z_{t-1}) in J_3. (We can assert this because we know that z_T is not in I_1.) Further, the number of periods in which z_t is in J_3 is either n or $n+1$. Specifically, it is $n+1$ if z_{T+N} is in J_3, and it is n if z_{T+N} is not in J_3. Thus, the remaining time periods (if any) in the set $\{T, T+1, ..., T+N\}$ must be spent in J_2. Thus, if $2n = N$ then the product $\prod_{t=T}^{T+N} v_t \leqslant D^{(N/2)+1}$ while if $2n < N$ then $\prod_{t=T}^{T+N} v_t \leqslant D^n\, D^{N+1-2n} < D^{(N/2)+1}$, so that in all cases

$$\prod_{t=T}^{T+N} v_t \leqslant D^{(N/2)+1}. \tag{4.4}$$

Thus, denoting $\prod_{t=0}^{T-1} v_t^{(1-\alpha)}$ by B, we have

$$\prod_{t=0}^{T+N} v_t^{(1-\alpha)} \leqslant B\, d^{(N/2)+1}. \tag{4.5}$$

Then, using (4.3), we see that (4.2) is satisfied.

5. A NUMERICAL EXAMPLE

In Section 4 we showed that if we can specify restrictions on the parameter values (of the endogenous growth model of Section 3) such that (E.0), (E.1), and (E.2) are simultaneously satisfied, then the endogenous growth model will exhibit chaotic equilibrium paths. We now specify numerical values of the parameters of the model under which (E.0), (E.1), and (E.2) can be verified.

Let $\alpha = 0.5$, $\beta = 5$, $\theta = 1.59$, $\delta = 0.6$, and $\mu = 1$. For these values of the parameters, it is easy to calculate that $(\beta+1)^\beta/\beta^\beta = (6/5)^5 = 2.48832$, $\delta^{1/(1-\alpha)}\theta^{\beta+[1/(1-\alpha)]} = \delta^2\theta^7 = 9.248735$, and $(\beta+1)^{(\beta+1)}/\beta^\beta = 6(6/5)^5 = 14.92992$. Thus, Condition 1 is clearly satisfied. Also, $\theta = 1.59 < 1.66 < (1/\delta)$, so that Condition 2 is also satisfied.

We now enumerate the growth factor of the steady state, $\bar{z}$. This is given by $\lambda = [1/(\delta\theta)^{0.4}]$. We can calculate that $\delta\theta = 0.954$ and $(\delta\theta)^{0.4} = 0.981339658$, so that $\lambda = 1.019015172$, which is greater than 1, as predicted by Condition 2.

We turn, now, to the modal point and its iterates under the map, f. The modal point, m, is $[\theta/(\beta+1)] = 0.265$. The parameter, A, in the map, f, is $(\delta\theta)^2 = 0.910116$. Thus, the first iterate of the modal point is $f(m) = Am(\theta - m)^5 = (0.910116)(0.265)(1.59 - 0.265)^5 = 0.984967758 \equiv m_1$.

The second iterate of the modal point $f^2(m) = f(m_1) = Am_1(\theta - m_1)^5 = 0.072679404 \equiv m_2$. And the third iterate of the modal point $f^3(m) = f(m_2) = Am_2(\theta - m_2)^5 = 0.531979469 \equiv m_3$. Note that $f^2(m) < m$ and $m < f^3(m) < z$, so (E.0) is satisfied.

It is straightforward to verify that (E.1) is satisfied. Given that $f^3(m) > 0.53$, we have $(\theta - f^3(m)) < 1.06$ and so $\delta^{1/(1-\alpha)}(\theta - f^3(m))^{[1/(1-\alpha)]+\beta} < (0.6)^2 (1.06)^7 = (0.36)(1.5036) = 0.541296 < 1$.

Verifying (E.2) is somewhat more involved. We will show that the function, h, from the set I to $\mathbb{R}_{++}$, defined by

$$h(y) = (\theta - y)(\theta - f(y)) \qquad \text{for} \quad y \quad \text{in} \quad I, \tag{5.1}$$

is bounded above by 1.3 on the set I_3.

Define $z^* = 0.59$. Then $f(z^*) = A(0.59) = 0.5789903$. Note that $z^* > \bar{z} = 0.5709848$ and $f(z^*) > f^3(m) = f(z')$, so that we must have $\bar{z} < z^* < z'$. Define $J = [z^*, f(m)]$; note that I_3 is a subset of J. We will, in fact, show that h is bounded above by 1.3 on the larger set J.

The reason for working with J is, of course, that the boundary points of this interval are easy to calculate. We can calculate $h(z^*) = (\theta - f(z^*)) = 1.05303156$. Also, at $y = f(m)$, we have $h(y) = (\theta - f(m))(\theta - f^2(m)) = (0.605032242)(1.517320596) = 0.918027882$. At $y = 0.69$, which is an interior point of J, we have $h(y) = (0.9)(\theta - f(y)) = (0.9)(1.219184066) = 1.09726566$.

We know that h attains a maximum on J. Denote by y^* the point at which h attains a maximum on J. By the above calculations, the maximum cannot be at a boundary point. Thus, we must have $h'(y^*) = 0$. This yields the equation

$$(\theta/A) = (\theta - y^*)^5 \, (7y^* - \theta). \tag{5.2}$$

Define $g(y) = (\theta - y)^5 \, (7y - \theta)$ for y in J. Then $g'(y) = (\theta - y)^4 \, (12\theta - 42y)$ for y in J, and since $(12\theta - 42y) \leqslant (12\theta - 42z^*) = 19.08 - 42(0.59) = 19.08 - 24.78 < 0$, $g'(y) < 0$ for all y in J. This means that (5.2) has a unique solution.

We now evaluate $h(y^*)$ as follows. We have $h(y^*) = (\theta - y^*)(\theta - f(y^*)) = (\theta - y^*)(\theta - Ay^*(\theta - y^*)^5) = (\theta - y^*)(\theta - Ay^*(\theta/[A(7y^* - \theta)])) = \theta(\theta - y^*)(6y^*)(6y^* - \theta)/(7y^* - \theta) \leqslant \theta(\theta - z^*)(6 - \theta)/(7 - \theta) = (1.59)(4.41)/(5.41) = 1.296099815 < 1.3$. Thus, $h(y) \leqslant 1.3$ on the set J and hence on the set I_3, as we had claimed.

We can now see that for all y in I_3, $\{\delta^{1/(1-\alpha)}(\theta - y)^{[1/(1-\alpha)]+\beta}\}\{\delta^{1/(1-\alpha)}(\theta - f(y))^{[1/(1-\alpha)]+\beta}\} = \delta^4 (h(y))^7 \leqslant (0.6)^4 \, (1.3)^7 = 0.8132207 < 1$, thus verifying (E.2).

Remarks (i) It is shown in [5] that the steady state, $\bar{z}$, will be locally unstable if

$$\delta^{[1/(1-\alpha)]}\theta^{\{\beta + [1/(1-\alpha)]\}} > (\beta + 2)^\beta / \beta^\beta. \tag{5.3}$$

We note that $(\beta + 2)^\beta/\beta^\beta = (7/5)^5 = 5.37824$, so (5.3) is satisfied in our example. The restriction (5.3) is called Condition 3 in [5].

(ii) We checked above that

$$f^2(m) < m \tag{5.4}$$

for our example. The restriction (5.4) is called Condition 4 in [5].

REFERENCES

1. R. L. Adler, A. G. Konheim, and M. H. McAndrew, Topological entropy, *Trans. Amer. Math. Soc.* **114** (1965), 309–319.
2. L. Alseda, J. Llibre, and M. Misiurewicz, "Combinatorial Dynamics and Entropy in Dimension One," World Scientific, Singapore, 1993.
3. L. Block, J. Guckenheimer, M. Misieurewicz, and L. S. Young, Periodic points and topological entropy of one dimensional maps, *in* "Global Theory of Dynamical Systems," Lecture Notes in Mathematics, Vol. 819, pp. 18–34, Springer-Verlag, Berlin, 1980.
4. L. S. Block and W. A. Coppel, "Dynamics in One Dimension," Lecture Notes in Mathematics, Vol. 1513, Springer-Verlag, Berlin, 1992.
5. M. Boldrin, K. Nishimura, T. Shigoka, and M. Yano, Chaotic equilibrium dynamics in endogenous growth models, mimeo 1999.
6. R. Bowen, Entropy for group endomorphisms and homogeneous spaces, *Trans. Amer. Math. Soc.* **153** (1971), 401–414.
7. R. Bowen, Periodic points and measures for Axiom *A* diffeomorphisms, *Trans. Amer. Math. Soc.* **154** (1971), 377–397.
8. A. Lasota and J. A. Yorke, On the existence on invariante measures for transformations with strictly turbulent trajectories, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Anstronom. Phys.* **25** (1977), 233–238.
9. T.-Y. Li and J. A. Yorke, Period three implies chaos, *Amer. Math. Monthly* **82** (1975), 985–992.
10. K. Matsuyama, Growing through cycles, *Econometrica* **67** (1999), 335–347.
11. M. Misieurewicz, Horseshoes for mappings of the interval, *Bull. Acad. Polon. Sci. Ser. Sci. Math.* **27** (1979), 167–169.
12. D. Ruelle and F. Takens, On the nature of turbulence, *Comm. Math. Phys.* **20** (1971), 167–192.
13. A. N. Sarkovskii, Coexistence of cycles of a continuous map of a line into itself, *Ukrainian Math. Zh.* **16** (1964), 61–71.